Graphs in -1 colors

Learning to Live with Stanley's Acyclicity Theorem

Oscar Coppola, Mikey Reilly

K&G June 25, 2021

The Main Result

Recall that a coloring of a graph G is a function κ which assigns every vertex in G a natural number. A coloring is proper if any two vertices which are connected by an edge have distinct colors.

The Main Result

Recall that a coloring of a graph G is a function κ which assigns every vertex in G a natural number. A coloring is proper if any two vertices which are connected by an edge have distinct colors.

$\chi_G(\lambda)$ gives the number of proper colorings of G in λ colors.

The Main Result

Recall that a coloring of a graph G is a function κ which assigns every vertex in G a natural number. A coloring is proper if any two vertices which are connected by an edge have distinct colors.

$\chi_G(\lambda)$ gives the number of proper colorings of G in λ colors.

Theorem (Stanley's Acyclicity Theorem)

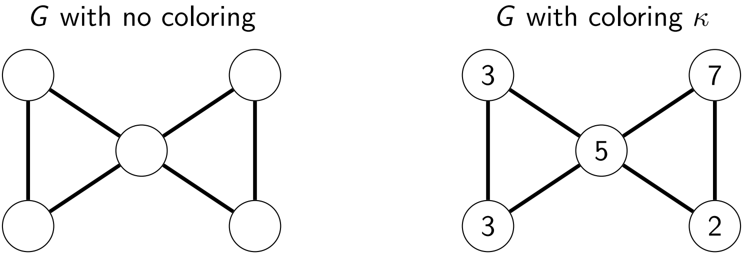
For any graph G with n vertices, the number of acyclic orientations of G is $(-1)^n \chi_G(-1)$, where χ_G is the chromatic polynomial of G.

Color by Numbers

To say that κ is a coloring of G in λ colors means that κ is a function $\kappa : \{\text{Vertices of } G\} \to \{1, 2, \ldots, \lambda\}$

Color by Numbers

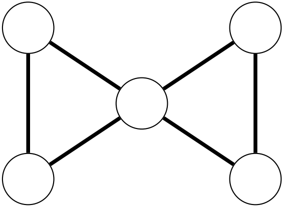
To say that κ is a coloring of G in λ colors means that κ is a function $\kappa : \{\text{Vertices of } G\} \to \{1, 2, \ldots, \lambda\}$



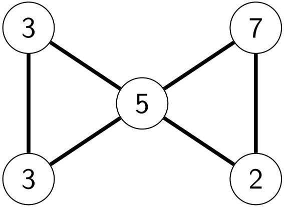
Color by Numbers

To say that κ is a coloring of G in λ colors means that κ is a function $\kappa : \{\text{Vertices of } G\} \to \{1, 2, \ldots, \lambda\}$

G with no coloring



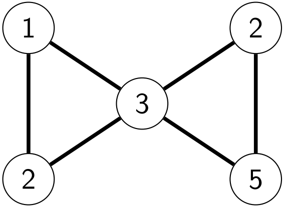
G with coloring κ



Note that κ is not a proper coloring since there are two vertices which are connected by an edge and have the same color.

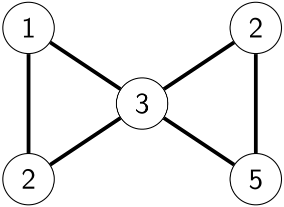
Coloring by Numbers (cont.)

G with proper coloring κ'



Coloring by Numbers (cont.)

G with proper coloring κ'



The chromatic polynomial for G is $\chi_G(\lambda) = \lambda^5 - 6\lambda^4 + 13\lambda^3 - 12\lambda^2 + 4\lambda$.

So for example, there are:

0 ways to properly color G in 2 colors

12 ways to properly color G in 3 colors

144 ways to properly color G in 4 colors

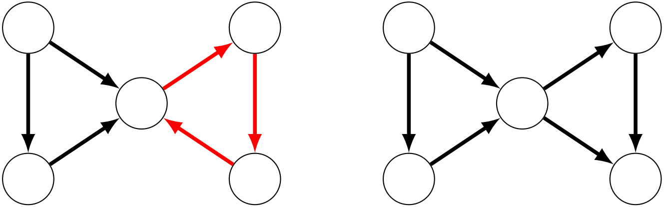
720 ways to properly color G in 5 colors

$\vdots$

What's an Acyclicity?

An orientation of a graph assigns a direction to each edge. The orientation is acyclic if it forms no cycles.

G with cyclic orientation σ G with acyclic orientation σ'



As it turns out, there is an interesting relation between graph colorings and orientations.

A Connection Between Colors and Directions

Given a proper coloring of your favorite graph G, we can create a unique orientation of G which will contain no cycles.

A Connection Between Colors and Directions

Given a proper coloring of your favorite graph G, we can create a unique orientation of G which will contain no cycles.

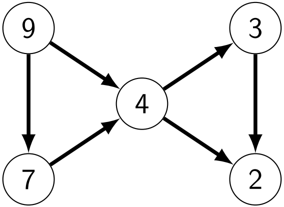
Specifically, each edge in G will point to the vertex whose coloring has a lower value.

A Connection Between Colors and Directions

Given a proper coloring of your favorite graph G, we can create a unique orientation of G which will contain no cycles.

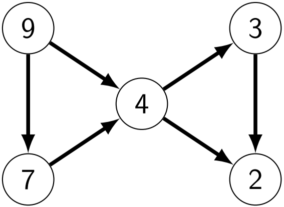
Specifically, each edge in G will point to the vertex whose coloring has a lower value.

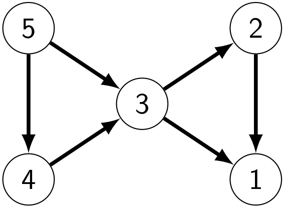
Any orientation created this way must be acyclic: the colors strictly decrease along each edge, so it would be impossible for a directed path to begin and end at the same vertex.



A Connection Between Colors and Directions (cont.)

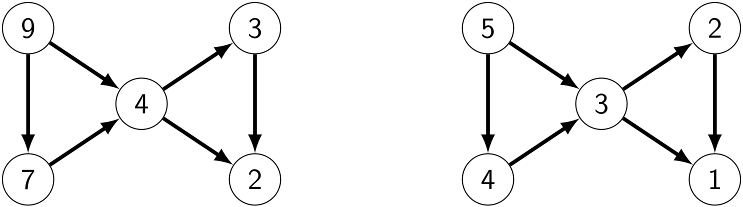
So, every proper λ-coloring of a graph G can be associated to an acyclic orientation of G. However, multiple proper colorings might be associated to the same orientation.





A Connection Between Colors and Directions (cont.)

So, every proper λ-coloring of a graph G can be associated to an acyclic orientation of G. However, multiple proper colorings might be associated to the same orientation.



Another way of phrasing this remark is that

$$\chi_G(\lambda) = \#\{(\kappa, \sigma) : \sigma \text{ is acyclic and } v_1 \xrightarrow{\sigma} v_2 \implies \kappa(v_1) > \kappa(v_2)\}$$

What's Upsilon?

Let's consider a similar function, υ_G, defined as

$$\upsilon_G(\lambda) = \#\{(\kappa, \sigma) : \sigma \text{ is acyclic and } v_1 \xrightarrow{\sigma} v_2 \implies \kappa(v_1) \geq \kappa(v_2)\}.$$

What's Upsilon?

Let's consider a similar function, υ_G, defined as

$$\upsilon_G(\lambda) = \#\{(\kappa, \sigma) : \sigma \text{ is acyclic and } v_1 \xrightarrow{\sigma} v_2 \implies \kappa(v_1) \geq \kappa(v_2)\}.$$

We call such pairs of colorings and orientations *compatible pairs*. Note that κ is no longer forced to be a proper coloring! In fact, κ could color the entire graph with the same color.

What's Upsilon?

Let's consider a similar function, υ_G, defined as

$$\upsilon_G(\lambda) = \#\{(\kappa, \sigma) : \sigma \text{ is acyclic and } v_1 \xrightarrow{\sigma} v_2 \implies \kappa(v_1) \geq \kappa(v_2)\}.$$

We call such pairs of colorings and orientations *compatible pairs*. Note that κ is no longer forced to be a proper coloring! In fact, κ could color the entire graph with the same color.

Specifically, we can see that $\upsilon_G(1)$ counts the number of acyclic orientations of G.

Our goal is to prove that $(-1)^n \upsilon_G(\lambda) = \chi_G(-\lambda)$, where n is the number of vertices in G.

Contracting and Deleting

Recall that χ is uniquely defined by the following conditions:

(1) $\chi_{\circ}(\lambda) = \lambda$

(2) $\chi_{G \sqcup H}(\lambda) = \chi_G(\lambda) \cdot \chi_H(\lambda)$

(3) $\chi_G(\lambda) = \chi_{G-e}(\lambda) - \chi_{G/e}(\lambda)$

Contracting and Deleting

Recall that χ is uniquely defined by the following conditions:

(1) $\chi_{\circ}(\lambda) = \lambda$

(2) $\chi_{G \sqcup H}(\lambda) = \chi_G(\lambda) \cdot \chi_H(\lambda)$

(3) $\chi_G(\lambda) = \chi_{G-e}(\lambda) - \chi_{G/e}(\lambda)$

We will show that υ has the following properties:

(1') $\upsilon_{\circ}(\lambda) = \lambda$

(2') $\upsilon_{G \sqcup H}(\lambda) = \upsilon_G(\lambda) \cdot \upsilon_H(\lambda)$

(3') $\upsilon_G(\lambda) = \upsilon_{G-e}(\lambda) + \upsilon_{G/e}(\lambda)$

Contracting and Deleting (cont.)

υ satisfies (1') and (2') for the same reason that χ does.

(1') $\upsilon_\circ(\lambda) = \lambda$

(2') $\upsilon_{G \sqcup H}(\lambda) = \upsilon_G(\lambda) \cdot \upsilon_H(\lambda)$

Contracting and Deleting (cont.)

υ satisfies (1') and (2') for the same reason that χ does.

(1') $\upsilon_\circ(\lambda) = \lambda$

(2') $\upsilon_{G \sqcup H}(\lambda) = \upsilon_G(\lambda) \cdot \upsilon_H(\lambda)$

First, if there are λ colors available, then there are λ ways to color a single vertex. For each coloring, there is only one way to "orient" a single vertex with no edges, so the total number of compatible pairs is λ.

Contracting and Deleting (cont.)

υ satisfies (1') and (2') for the same reason that χ does.

(1') $\upsilon_\circ(\lambda) = \lambda$

(2') $\upsilon_{G \sqcup H}(\lambda) = \upsilon_G(\lambda) \cdot \upsilon_H(\lambda)$

First, if there are λ colors available, then there are λ ways to color a single vertex. For each coloring, there is only one way to "orient" a single vertex with no edges, so the total number of compatible pairs is λ.

Second, if G and H are disjoint graphs, then any compatible pair for G put together with a compatible pair for H will make a compatible pair for $G \sqcup H$. Additionally, a compatible pair for $G \sqcup H$ will remain a compatible pair when restricted to either G or H. So, $\upsilon_{G \sqcup H}(\lambda) = \upsilon_G(\lambda) \cdot \upsilon_H(\lambda)$.

Contracting and Deleting (cont.)

Now to show that $v_G(\lambda) = v_{G-e}(\lambda) + v_{G/e}(\lambda)$.

Contracting and Deleting (cont.)

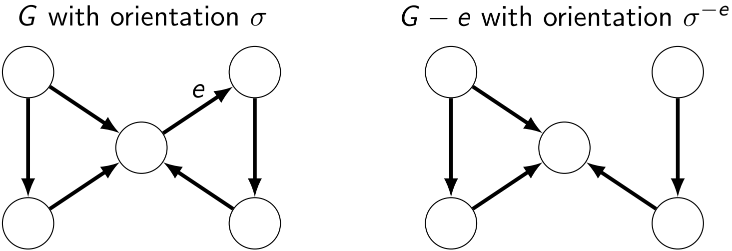
Now to show that $\upsilon_G(\lambda) = \upsilon_{G-e}(\lambda) + \upsilon_{G/e}(\lambda)$.

Pick your favorite edge e in G. Consider σ^{-e} to be the orientation on $G - e$ that agrees with σ on all edges other than e (it has no direction on e since e is not an edge in $G - e$.)

Contracting and Deleting (cont.)

Now to show that $\upsilon_G(\lambda) = \upsilon_{G-e}(\lambda) + \upsilon_{G/e}(\lambda)$.

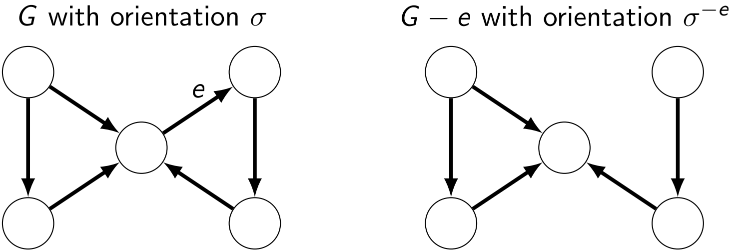
Pick your favorite edge e in G. Consider σ^{-e} to be the orientation on $G - e$ that agrees with σ on all edges other than e (it has no direction on e since e is not an edge in $G - e$.)



Contracting and Deleting (cont.)

Now to show that $\upsilon_G(\lambda) = \upsilon_{G-e}(\lambda) + \upsilon_{G/e}(\lambda)$.

Pick your favorite edge e in G. Consider σ^{-e} to be the orientation on $G - e$ that agrees with σ on all edges other than e (it has no direction on e since e is not an edge in $G - e$.)



We will first show that that the mapping $(\kappa, \sigma) \mapsto (\kappa, \sigma^{-e})$ between compatible pairs for G and compatible pairs for $G - e$ is surjective for any choice of edge e.

Proving Surjectivity

Lemma

For any compatible pair (κ, ω) *for* $G - e$, ω *can be extended to some* ω' *on* G *such that* (κ, ω') *is a compatible pair for* G.

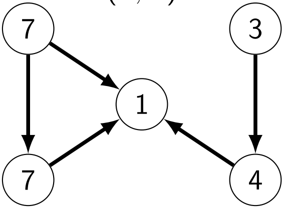
Proving Surjectivity

Lemma

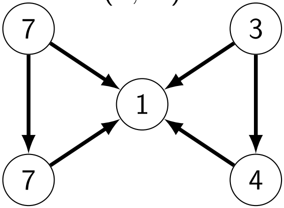
For any compatible pair (κ, ω) for $G - e$, ω can be extended to some ω' on G such that (κ, ω') is a compatible pair for G.

If u and v are the endpoints of e with $\kappa(u) \neq \kappa(v)$, then defining ω' to point toward the vertex with the smaller color will result in (κ, ω') being a compatible pair.

(κ, ω)



(κ, ω')



Surjectivity, Subjectively

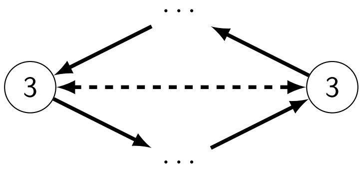
If $\kappa(u) = \kappa(v)$, then it is impossible for both choices of direction for e to result in a cycle, so we may pick whichever one makes ω' acyclic.

Surjectivity, Subjectively

If $\kappa(u) = \kappa(v)$, then it is impossible for both choices of direction for e to result in a cycle, so we may pick whichever one makes ω' acyclic.

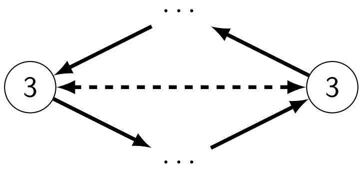
Surjectivity, Subjectively (cont.)

How do we know that one of them will be acyclic?



Surjectivity, Subjectively (cont.)

How do we know that one of them will be acyclic?



The upshot of all of this is that the mapping $(\kappa, \sigma) \mapsto (\kappa, \sigma^{-e})$ between compatible pairs for G and compatible pairs for $G - e$ is surjective for any choice of edge e.

Injectivity, Objectively

If this mapping is a bijection, then we are done.

So is this map injective?

Injectivity, Objectively

If this mapping is a bijection, then we are done.

So is this map injective? No.

Injectivity, Objectively

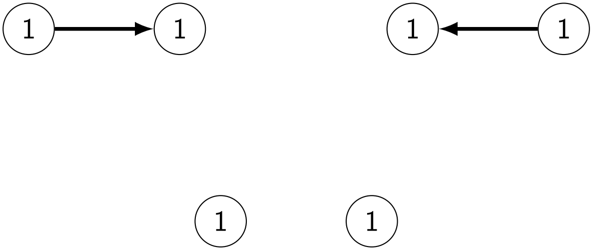
If this mapping is a bijection, then we are done.

So is this map injective? No.

Injectivity, Objectively

If this mapping is a bijection, then we are done.

So is this map injective? No.



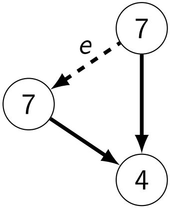
Injectivity, Objectively (cont.)

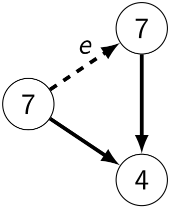
Just how non-injective is this map?

Injectivity, Objectively (cont.)

Just how non-injective is this map?

For two compatible pairs for G to be mapped to the same compatible pair for $G - e$, they must agree on every edge except for e. If e can point in either direction, then it must be that $\kappa(u) = \kappa(v)$ since $\kappa(u) \geq \kappa(v)$ and $\kappa(u) \leq \kappa(v)$ (where u and v are the endpoints of e).

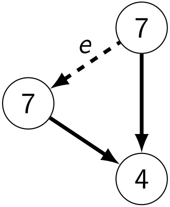


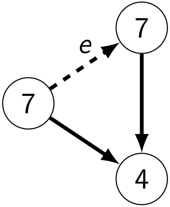


Injectivity, Objectively (cont.)

Just how non-injective is this map?

For two compatible pairs for G to be mapped to the same compatible pair for $G - e$, they must agree on every edge except for e. If e can point in either direction, then it must be that $\kappa(u) = \kappa(v)$ since $\kappa(u) \geq \kappa(v)$ and $\kappa(u) \leq \kappa(v)$ (where u and v are the endpoints of e).

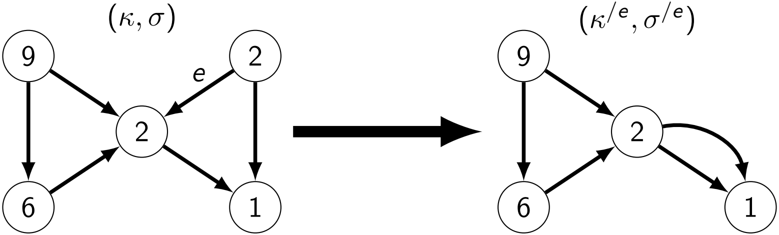




However, this means that a compatible pair is induced on G/e.

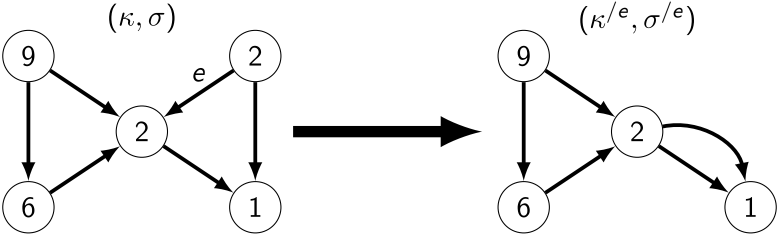
Contractually Obligated Injectivity

If $\kappa(u) = \kappa(v)$, then we can easily define $\kappa^{/e}$ and $\sigma^{/e}$ on G/e. We demand that $\kappa^{/e}$ and $\sigma^{/e}$ agree with κ and σ wherever possible and $\kappa^{/e}(w) = \kappa(u) = \kappa(v)$ where w is the vertex in G/e created by fusing u and v together.



Contractually Obligated Injectivity

If $\kappa(u) = \kappa(v)$, then we can easily define $\kappa^{/e}$ and $\sigma^{/e}$ on G/e. We demand that $\kappa^{/e}$ and $\sigma^{/e}$ agree with κ and σ wherever possible and $\kappa^{/e}(w) = \kappa(u) = \kappa(v)$ where w is the vertex in G/e created by fusing u and v together.



Therefore, any compatible pair for G/e can be extended to not just one compatible pair for G, but in fact two compatible pairs for G.

Putting it All Together

Considering all of the compatible pairs for G (which by definition is $\upsilon_G(\lambda)$), we've seen that they can all be produced by extending the $\upsilon_{G-e}(\lambda)$ compatible pairs of $G - e$, and that $\upsilon_{G/e}(\lambda)$ of those pairs admit either direction of e in the extension.

Putting it All Together

Considering all of the compatible pairs for G (which by definition is $\upsilon_G(\lambda)$), we've seen that they can all be produced by extending the $\upsilon_{G-e}(\lambda)$ compatible pairs of $G - e$, and that $\upsilon_{G/e}(\lambda)$ of those pairs admit either direction of e in the extension.

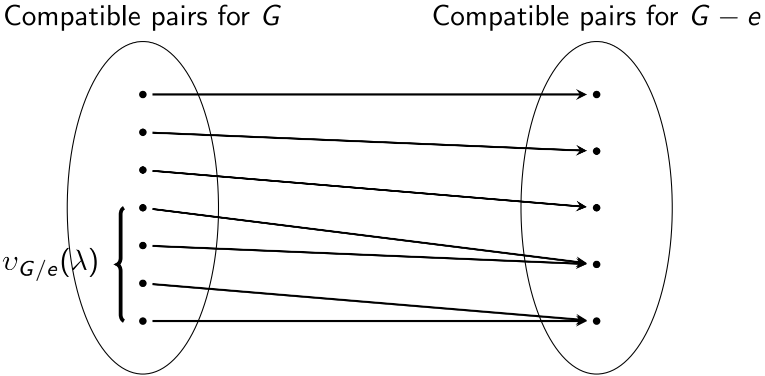
So we may consider our map from before that maps compatible pairs of G to compatible pairs for $G - e$, $(\kappa, \sigma) \mapsto (\kappa, \sigma^{-e})$. We have found that there are two pairs which are mapped to the same thing under this map for each compatible pair for G/e (of which there are $\upsilon_{G/e}(\lambda)$) and moreover, these are the only pairs which are mapped to the same thing.

Putting it All Together (cont.)

So, we can put some of the compatible pairs for G in 1-to-1 correspondence with the compatible pairs for $G - e$ (of which there are $\upsilon_{G-e}(\lambda)$) and have $\upsilon_{G/e}(\lambda)$ compatible pairs left over.

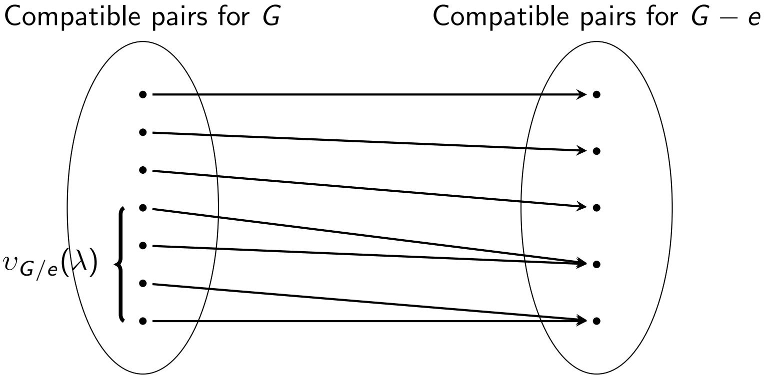
Putting it All Together (cont.)

So, we can put some of the compatible pairs for G in 1-to-1 correspondence with the compatible pairs for $G - e$ (of which there are $\upsilon_{G-e}(\lambda)$) and have $\upsilon_{G/e}(\lambda)$ compatible pairs left over.



Putting it All Together (cont.)

So, we can put some of the compatible pairs for G in 1-to-1 correspondence with the compatible pairs for $G - e$ (of which there are $\upsilon_{G-e}(\lambda)$) and have $\upsilon_{G/e}(\lambda)$ compatible pairs left over.



In other words, we have that $\upsilon_G(\lambda) = \upsilon_{G-e}(\lambda) + \upsilon_{G/e}(\lambda)$.

A Negative Number of Colors

Using this information, we can show that $(-1)^n \upsilon_G(\lambda) = \chi_G(-\lambda)$.

A Negative Number of Colors

Using this information, we can show that $(-1)^n \upsilon_G(\lambda) = \chi_G(-\lambda)$.

We have that

$$(-1)^1 \upsilon_\circ(\lambda) = -\lambda \tag{1'}$$

$$(-1)^{n+m} \upsilon_{G \sqcup H}(\lambda) = (-1)^n \upsilon_G(\lambda) \cdot (-1)^m \upsilon_H(\lambda) \tag{2'}$$

A Negative Number of Colors

Using this information, we can show that $(-1)^n \upsilon_G(\lambda) = \chi_G(-\lambda)$.

We have that

$$(-1)^1 \upsilon_\circ(\lambda) = -\lambda \tag{1'}$$

$$(-1)^{n+m} \upsilon_{G \sqcup H}(\lambda) = (-1)^n \upsilon_G(\lambda) \cdot (-1)^m \upsilon_H(\lambda) \tag{2'}$$

And lastly,

$$\begin{aligned}(-1)^n \upsilon_G(\lambda) &= (-1)^n \upsilon_{G-e}(\lambda) + (-1)^n \upsilon_{G/e}(\lambda) \\ &= (-1)^n \upsilon_{G-e}(\lambda) - (-1)^{n-1} \upsilon_{G/e}(\lambda) \qquad (3')\end{aligned}$$

A Negative Number of Colors (cont.)

We know that $\chi_G(-\lambda)$ is uniquely defined by the following properties

(1) $\chi_{\circ}(-\lambda) = -\lambda$.

(2) $\chi_{G \sqcup H}(-\lambda) = \chi_G(-\lambda) \cdot \chi_H(-\lambda)$.

(3) $\chi_G(-\lambda) = \chi_{G-e}(-\lambda) - \chi_{G/e}(-\lambda)$.

We have shown that $(-1)^n \upsilon_G(\lambda)$ satisfies each of these properties and so we have that

$$(-1)^n \upsilon_G(\lambda) = \chi_G(-\lambda)$$

QED ■

Using this formula, we can note that by plugging in $\lambda = 1$ we arrive at

$$(-1)^n \upsilon_G(1) = \chi_G(-1)$$

$\upsilon_G(1)$ is the number of acyclic orientations of G since there is only one way to color a graph in one color.

QED ■

Using this formula, we can note that by plugging in $\lambda = 1$ we arrive at

$$(-1)^n \upsilon_G(1) = \chi_G(-1)$$

$\upsilon_G(1)$ is the number of acyclic orientations of G since there is only one way to color a graph in one color.

And so in conclusion:

QED ■

Using this formula, we can note that by plugging in $\lambda = 1$ we arrive at

$$(-1)^n \upsilon_G(1) = \chi_G(-1)$$

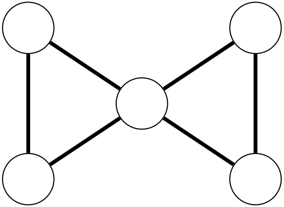
$\upsilon_G(1)$ is the number of acyclic orientations of G since there is only one way to color a graph in one color.

And so in conclusion:

The number of acyclic orientations of $G = \upsilon_G(1) = (-1)^n \chi(-1)$

Example

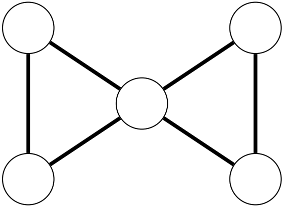
Consider our favorite graph G:



It has 6 edges and so it has $2^6 = 64$ orientations. There are only two paths that can be cycles. If one of the two paths is a cycle, then there are $2^3 = 8$ ways to orient the other half. There are two possible cycles and each cycle can be directed in one of two ways and so (taking into account the 4 ways in which there can be two cycles) we have that there are $8 \cdot 2 \cdot 2 - 4 = 28$ orientations of G which are cyclic.

Example

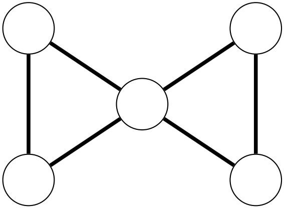
Consider our favorite graph G:



It has 6 edges and so it has $2^6 = 64$ orientations. There are only two paths that can be cycles. If one of the two paths is a cycle, then there are $2^3 = 8$ ways to orient the other half. There are two possible cycles and each cycle can be directed in one of two ways and so (taking into account the 4 ways in which there can be two cycles) we have that there are $8 \cdot 2 \cdot 2 - 4 = 28$ orientations of G which are cyclic.

Therefore G has $64 - 28 = 36$ acyclic orientations.

Example (cont.)



Recall that the chromatic polynomial for G is
$\chi_G(\lambda) = \lambda^5 - 6\lambda^4 + 13\lambda^3 - 12\lambda^2 + 4\lambda$.

G has 5 vertices and so Stanley's Acyclicity Theorem tells us that G has $(-1)^5\chi_G(-1) = -(-1-6-13-12-4) = 36$ acyclic orientations.

Source

Stanley, Richard P. “Acyclic Orientations of Graphs.” Discrete Mathematics, vol. 5, no. 2, 1973, pp. 171–178., doi:10.1016/0012-365x(73)90108-8.